Experiments with Mercury Jets.

By S. W. J. Smith, M.A., D.Sc., F.R.S., and H. Moss, M.Sc., Imperial College, South Kensington.

(Received February 22, 1917.)

Section I.—*The Relation between the Jet-length and the Velocity of Efflux.*

1. *Introductory.*—Certain electro-chemical experiments, to be described in a subsequent communication, arose out of peculiarities which we noticed in the behaviour of a stream of mercury issuing from the lower, drawn-out, end of a vertical tube containing that liquid. The salient feature of the phenomena is the relation between the length of the continuous part of the liquid stream, called for convenience the "jet-length," and the "head" under which the mercury is ejected.

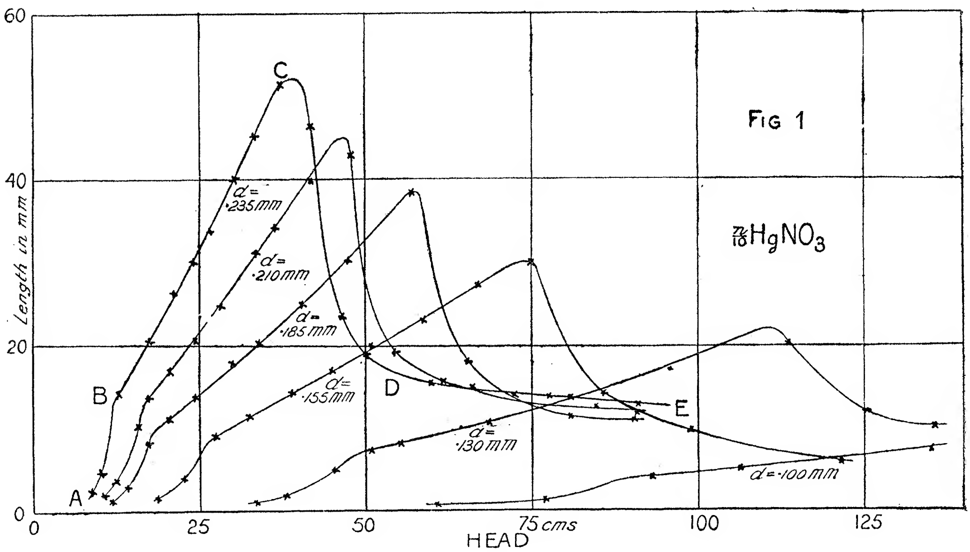
The experiments were made with orifices of diameters varying from about 0·25 mm. downwards to about 0·1 mm. It was found that the relation between the jet-length l and the head h depended appreciably upon the form of the nozzle through which the mercury escaped. For definiteness the nozzles were all constructed in the same way, namely by drawing out a piece of capillary tubing of about 1 mm. bore and severing it, as nearly as possible perpendicularly, at the centre of the constricted part. The nozzle so obtained tapered, in between 1 and 2 cm., from the original diameter of the tube to the diameter of the orifice and was very nearly cylindrical at the point of discharge. The values of the jet-length were determined by means of a cathetometer microscope.

2. *General Features of the lh Curves for Jets of Different Diameters and Constant Surface Tension.*—In the particular experiments represented in fig. 1, the mercury fell into a solution of mercurous nitrate (of strength 0·1 grm.-mol. per litre) contained in a vessel with flat glass sides. The object of using this solution was to ensure that the surface tension of the mercury should be practically constant throughout the measurements.

Considering the curve ABCDE, of the figure, which refers to an orifice of diameter 0·235 mm., it will be seen that between A and B, where the heads are at first only just sufficient to cause the jet to form, the jet-length increases very rapidly as the head rises. At B the rate of increase of l with h changes suddenly, becoming smaller and remaining practically constant till C is reached. Beyond this point the jet-length falls, very rapidly at first, as the head is increased. At the higher heads (near E) the rate of decrease of jet-length is comparatively small. Between C and D the jet is relatively unstable,

lengthening and shortening in a capricious way which makes definite measurements of its length difficult. Sometimes it is impossible to obtain anything more than rough values of its mean length.

It will be seen that the other jets give *lh* curves of the same character as



that just described. In each case there are two critical points at which the value of dl/dh changes suddenly. The critical jet-lengths decrease and the critical heads rise as the diameter of the orifice diminishes.

3. *The lh Curves for Jets of Constant Diameter and Different Surface Tensions.*—In order to find how the positions of the critical points depended upon the surface tension we made the observations represented in fig. 2 with an orifice of 0·21 mm. diameter.

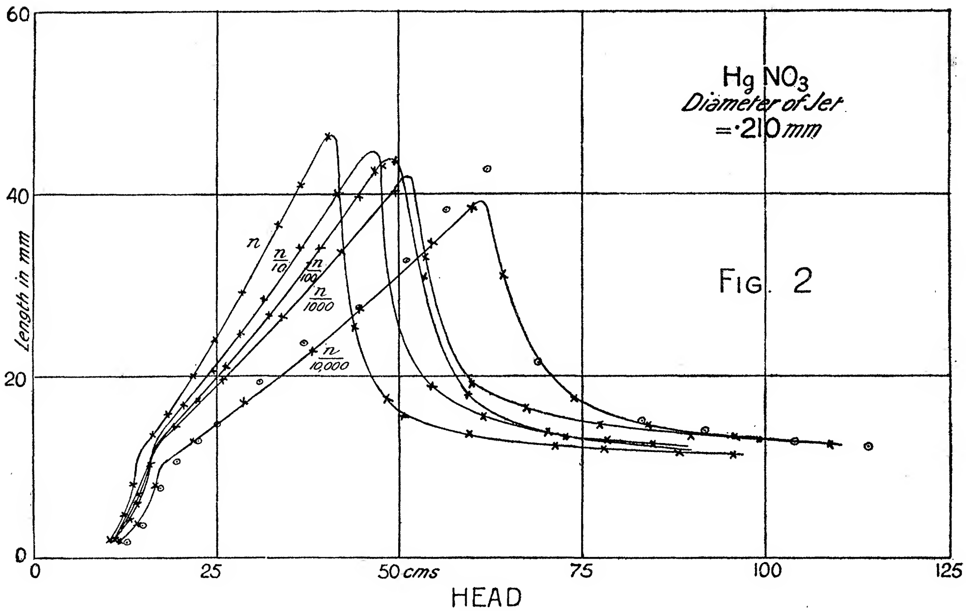
The different curves were obtained using solutions of $HgNO_3$ of different strengths, it being known that the surface tension between mercury and a solution of $HgNO_3$ decreases as the salt-content rises. The points marked by circles (⊙) were obtained when distilled water was used. It will be noticed at once that, although the critical jet-length does not alter greatly, the critical head rises as the surface tension increases.

4. *Critical Velocities of Efflux.*—It is sufficient for the experiments referred to in §1 that the phenomena exhibited in figs. 1 and 2 exist; but they are of some general interest and we have therefore endeavoured to verify certain impressions as to their cause.

The sudden drop in the jet-length at the higher critical point reminded us of the experiments of Osborne Reynolds dealing with the critical velocity of

flow of a liquid through a pipe. It seemed likely that there might be similar critical velocities for jets of the kind we used.

In order to test this impression we performed some qualitative experiments with a water jet into which, along its axis, was fed a thin column of red ink.



This jet gave lh variations resembling those of figs. 1 and 2 and the ink remained a thin stream along the axis until the upper critical head was reached. Immediately after this was passed, turbulent motion became obvious. The ink was dispersed throughout the jet.

We could not, of course, perform similar experiments with the mercury; but, although the water jet was comparatively very wide, it seemed reasonable to suppose that the higher critical point had the same cause in both cases. We were, therefore, led to regard the peculiar behaviour of the jets as due to the existence of critical velocities. From this point of view it seemed likely that the lower critical velocity was the counterpart of the phenomenon observed by Allen* in connection with the motion of spheres through liquids.

5. *Theoretical Relation between the Variables upon which Critical Velocities Depend.*—If, in the case of a pipe, we suppose the only variables involved to be its diameter d and the viscosity μ and density ρ of the liquid, we can (as

* 'Phil. Mag.,' September, 1900.

is well known) deduce a relation between the critical velocity and the variables by the method of dimensions. The result is

$$v_c \propto \mu (\rho d)^{-1}$$

where v_c is the critical velocity.

In the case of a jet, however, we have a free surface and therefore an additional variable, the surface tension of the liquid, upon which the critical velocities may depend. Introducing this factor into the dimensional equation we now get

$$v_c \propto \sigma^x \mu^{1-2x} (\rho d)^{x-1},$$

where σ is the surface tension and x is an undetermined index.

If we accept this result we see that x must be positive since, from fig. 2, we find that v_c increases with σ. Also, since, from fig. 1, v_c increases as d diminishes, $x-1$ must be negative. Therefore the value of x must lie between zero and unity.

This is as far as we can go without further experimental data, unless, for example, we assume that the viscosity, if it has any influence, must tend to increase the critical velocity. In that case we might draw the further conclusion that $1-2x$ must be positive and therefore that x cannot exceed $\frac{1}{2}$.

6. *Experimental Relation between the Critical Velocity and the Diameter of the Jet.*—The velocities of the jet were not determined in the experiments of fig. 1; but in another set of measurements (in which the mercury fell into distilled water) the upper critical velocities were determined approximately. The following Table contains the results :—

Diameter of orifice. d.	Critical head. h.	Critical velocity. v.	$\frac{v\sqrt{d}}{1000}$.
cm.	cm.	cm./sec.	
0·0127	111·5	230	25·9
0·0172	77	223	29
0·0202	54	188	26·7
0·0248	42	170	26·8
0·0295	39	145	25

Measurements of this kind cannot be made very accurately because very slight disturbances are sufficient to alter considerably the head under which the jet breaks down and it is of course impossible to guarantee that the orifices are of strictly similar character. But the experiments obviously suggest that the critical velocity is inversely proportional to the square root of the diameter, *i.e.*, that $x = \frac{1}{2}$.

7. *Experimental Relation between the Critical Velocity and the Surface Tension.*—Another approximate estimate of x can be made by means of the

results shown in fig. 2. The ratio of the surface tensions of the n and $n/1000$ solutions of $HgNO_3$ was measured and found to be nearly 1·3. The upper critical velocities for these solutions were not determined at the time, but from subsequent observations it can be inferred that their ratio was not far from 1·15. Hence, since $\sqrt{1·3} = 1·14$, the value $x = \frac{1}{2}$ again comes into view.

8. *The Physical Significance of the Results.*—These results suggest that the viscosity of the liquid has little influence upon the critical velocities and that, to a first approximation at any rate, we may write

$$v_c \propto (\sigma/\rho d)^{\frac{1}{2}}.$$

If we put this in the form

$$\rho v_c^2 \propto \sigma/d$$

we see at once, without recourse to the method of § 5, that it is dimensionally correct. For, as is well known, both quantities have the dimensions of pressure. The first term suggests the operation of "inertia" forces; the second the opposing influence of the pressure due to surface tension. We return to this later on. We may, however, remark here that we should not expect the relation to be universally true. Its simplicity must arise from the fact that, in particular cases, the "inertia" forces are predominant. There are probably other cases in which, for example when the viscosity is large, "viscous" forces are no longer negligible.

9. *Lord Rayleigh's Theory of Liquid Jets.*—The quantities called "critical velocities" in the preceding paragraphs are not accompanied by any perceptible discontinuities in the relation between the velocity of efflux and the head; the changes in the jet-length are the only indication of their existence. We therefore turned our attention next to the consideration of the factors which determine the length of the jet, and have endeavoured to apply Lord Rayleigh's theory to the results which we have obtained. This theory* develops the consequences of the fact, first noticed by Plateau and applied by him to the explanation of Savart's results, that a cylindrical liquid column is unstable when its length exceeds its circumference. It follows from this fact that any disturbance impressed upon the liquid column at the orifice will, if its wave-length exceeds the circumference of the column, increase in amplitude as it travels downwards until, finally, the sides of the column will come together and the continuity of the jet will be destroyed. The time which elapses before this occurs will depend upon the initial amplitude of the disturbance and upon the rate at which that amplitude grows.

* 'Proc. Lond. Math. Soc.,' vol. 10, p. 4 (1878).

If a is the initial amplitude, a_0 the value which it has to acquire before disintegration occurs, and t the time required, we may suppose these connected by an equation of the form

$$a_0 = ae^{qt}$$

or
$$t = \frac{1}{q}\log_e a_0/a,$$

Whence, if we knew the value of q and of a for any given disturbance, we could calculate the value, $l = vt$, which the jet-length would have for a given velocity v (supposed constant) of descent of the liquid.

It is easy to show by the method of dimensions that, if viscosity be neglected, we must have

$$q \propto 1/t \propto \sqrt{(\sigma/\rho d^3)}.$$

Lord Rayleigh has further shown that the exact result is of the form

$$q = f(x) \cdot \sqrt{(\sigma/\rho d^3)},$$

where $x = \lambda/\pi d$ is the ratio of the wave-length of the disturbance to the circumference of the jet, and that q has a well-defined maximum value occurring near $x = 1{\cdot}44$.

For equal amplitudes, therefore, a disturbance of wave-length $\lambda = 1{\cdot}44 \times \pi d$ will cause disintegration more rapidly than any other. Moreover, if we considered disturbances of unequal initial amplitudes a_1 and a_2, for which the values of q were q_1 and q_2, the times t_1 and t_2 required by them to produce disintegration would be connected by the equation

$$t_2 = t_1 + (q_1 - q_2)\, t_1/q_2 - 1/q_2 \cdot \log_e a_2/a_1$$

whence, for $q_1 > q_2$, we should have $t_2 > t_1$ even with $a_2 > a_1$ if we had $(q_1 - q_2)\, t_1 > \log_e a_2/a_1$.

Hence, if the times of disintegration were long enough, the disturbance with the greatest q would produce disintegration sooner than any other.

Without exact knowledge of the initial disturbance of the jet, we can imagine it resolved into components of different wave-lengths and amplitudes, whose effects are superposed, and can suppose that the relation between the wave-lengths and amplitudes of these components determines the conditions under which the jet breaks down.

The time of disintegration will be given by

$$a_0 = ae^{ty\sqrt{(\sigma/\rho d^3)}},$$

where a is the initial amplitude of the disintegrating disturbance, and y is the value of $f(x)$ with which its wave-length corresponds.

Substituting l/v for t we get

$$l = v \cdot 1/y \cdot \log a_0/a \cdot \sqrt{(\rho d^3/\sigma)}$$

as the simplest theoretical relation between the variables with which we are concerned.

10. *The Difference between our Results and those of Savart in Terms of this Theory.*—For our jets, unlike those of Savart, in which the liquid (water) flowed through a hole, several millimetres in diameter, in the bottom of a containing vessel, the value of l/v was not even approximately constant.

We may take this to mean that, in our case, the quantity $1/y \cdot \log a_0/a$ was variable and may assume that the variation was probably confined to a.

The experiments of which the results are collected in fig. 3 were made, primarily, to test more closely the truth of the relation

$$v_c \propto \sqrt{(\sigma/d)}$$

suggested by the experiments of figs. 1 and 2.

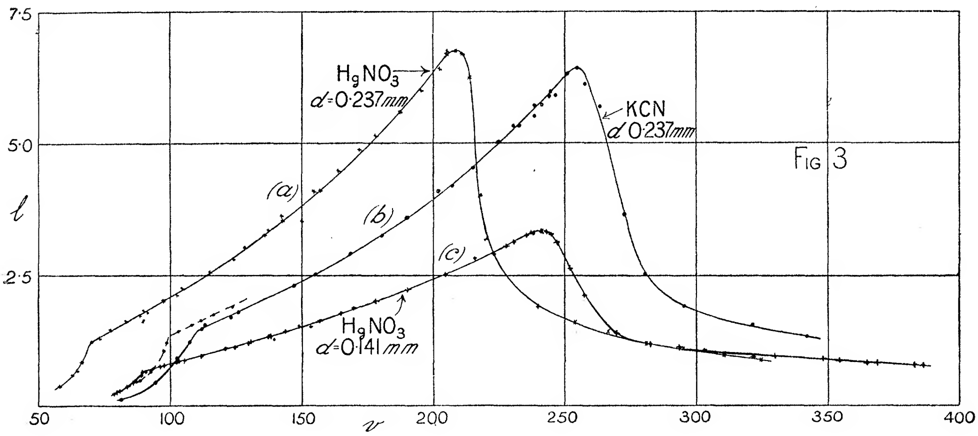


Fig 3

In this figure the values of l (in cm.), for certain cases, are plotted against the values of v (in cm. per sec.). The latter were determined by weighing the amounts of mercury which escaped through the nozzle in measured times, and by dividing the volumes escaping per second by the cross-sectional area of the orifice. The error introduced by this method of calculation is not great because, in consequence of their shortness in comparison with their initial velocity, our jets (unlike those of Savart which were very much longer) were nearly cylindrical. Except at the lowest velocities they did not taper perceptibly. The calculated velocity probably represents, within one or two per cent., the mean velocity of the jet.

In experiments (a) and (b) the diameter of the jet was 0·2365 mm.; in experiment (c) it was 0·141 mm. In (a) and (c) the solution was $n/4$ $HgNO_3$; in (b) it was $n/4$ KCN.

The curves are, of course, of the same general character as the lh curves of figs. 1 and 2.

11. *Application of the Theory to Account for the Form of the lv Curve.*—From the final expression in § 9, assuming y constant, we get

$$\frac{dl}{dv} = \frac{1}{y} \cdot \sqrt{\frac{\rho d^3}{\sigma}} \left\{ \log \frac{a_0}{a} - \frac{v}{a} \cdot \frac{da}{dv} \right\}.$$

From this we see that the form of the first part of the lv curve can be accounted for if we assume da/dv to be negative when v is small, *i.e.*, if we suppose the initial amplitude of the disintegrating disturbance decreases at first as the velocity of efflux rises.

We can explain the second part if the value of da/dv changes suddenly at the first critical point. Just beyond this point dl/dv is seen to be roughly equal to l/v. The interpretation of this would be that here da/dv is practically zero. At higher velocities it becomes appreciably negative again and remains of this sign until the second critical point is reached. Here, in order to explain the third part of the curve, we have to assume that da/dv acquires, comparatively suddenly, a large positive value.

We could form a picture of this sequence of events by imagining that the incipient turbulence appearing at the first critical point is just sufficient, at that point, to counteract the tendency of the amplitude to decrease as the velocity rises. The fuller turbulence appearing at the higher critical point is, on the other hand, great enough to cause a large increase in the amplitude of the initial disturbance and a corresponding decrease in the length of the jet.

12. *More Accurate Determination of the Relation between the Form of the lv Curve and the Value of the Surface Tension.*—The experiments (a) and (b) were to be used to test whether the relation

$$v_1/v_2 = \sqrt{(\sigma_1/\sigma_2)}$$

connects the critical velocities v_1 and v_2 when σ varies and d is constant.

We chose solutions of $HgNO_3$ and of KCN for our purpose because it was desirable that the surface tensions should be (1) easily comparable, (2) as different as possible, and (3) as free as possible from time effects.

A satisfactory solution giving a low value for σ is easily found by using $HgNO_3$; but it is difficult to find a correspondingly satisfactory solution for which σ is high. From previous experience we knew that undesirable time effects, which are generally present (when solutions other than those of mercury salts are employed), might be relatively small if we used $n/4$ KCN.* We knew, however, that, in general, there would be a time effect, in solutions

* Smith and Moss, 'Phil. Mag.,' April, 1908, p. 484.

of KCN, tending to make the surface tension decrease as the time of contact lengthened.

13. *Difficulties due to Variability of Surface Tension.*—When solutions of $HgNO_3$, for which the time effect is known to be very small, are employed, the second branch BC of the *lh* curve (*cf.* fig. 1) is very nearly linear. Hence, it is presumable that the surface tension between any other solution and mercury is not constant if the corresponding branch of its *lh* curve does not exhibit the same peculiarity.

Fig. 4 shows the *lh* curves for the solutions (*a*) and (*b*) of fig. 3.

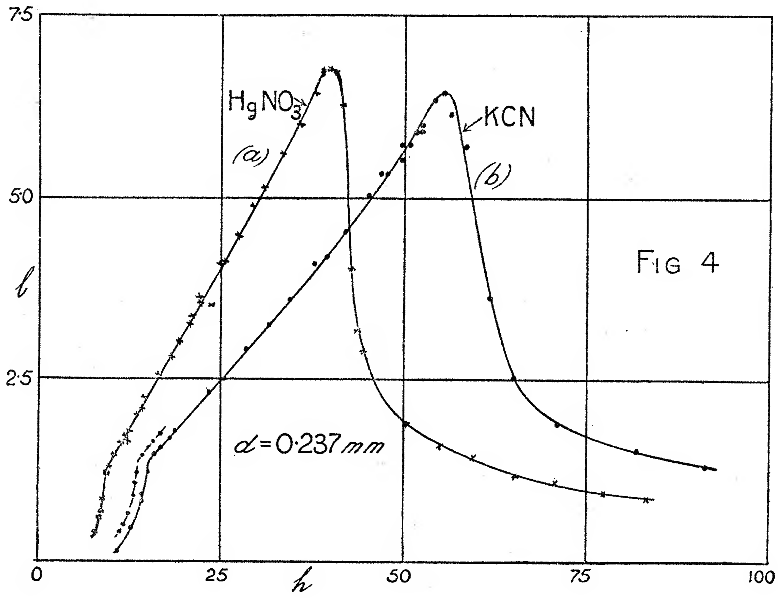


FIG 4

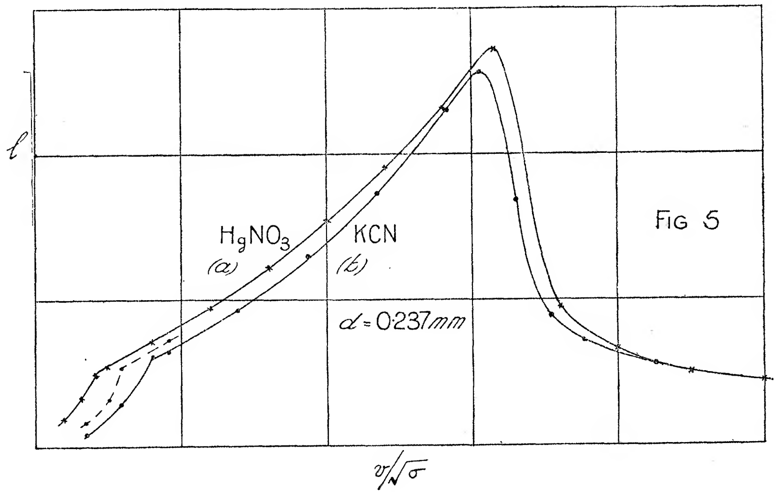
As will be seen, the second branch of the $HgNO_3$ curve is very nearly linear, while that of the KCN curve bends upwards as *h* rises. This difference suggests that the surface tension between the Hg and the KCN was not constant, as it might have been if we had hit upon the right concentration. By comparison with fig. 2, we see that (as was to be expected if it varied) the surface tension probably decreased as the time of contact lengthened.

14. *Verification of the Preliminary Inference of* § 7.—The ratio of the surface tensions of the two solutions was measured by the capillary tube method and hence corresponded with comparatively long times of contact between the mercury and the solutions.

At the second critical points of the *lv* curves the times of contact are relatively long. We should therefore expect to get a more favourable test of the relation $v_1/v_2 = \sqrt{(\sigma_1/\sigma_2)}$ from the observations at these points than

from those at the lower critical points, where the times of contact are shorter.

The observed value of σ_1/σ_2 was 1·57; whence, if the suggested relation were true, we should have $v_1/v_2 = 1{\cdot}25$. In fig. 5 the lv curve for the $HgNO_3$ is repeated from fig. 3, but that for the KCN has been obtained by leaving the ordinates unchanged and reducing all the abscissæ in the ratio 1·25 : 1.



It will be seen that the relation $v_1/v_2 = \surd(\sigma_1/\sigma_2)$ holds, within the limits of experimental error, between the upper critical velocities.

Comparing the lower critical velocities, however, we see that v_1/v_2 is greater than 1·25. This is what we should expect if, as is likely, σ_1/σ_2 here exceeds 1·57.

A repetition of the KCN measurements at the lower velocities with a solution which had stood for some time in contact with mercury gave the dotted curve shown in the figure. It is probable that in this case the surface tensions differed less from those at the higher velocities and, as is seen, the curve agrees more nearly with that for the $HgNO_3$ than the corresponding part of the KCN curve obtained earlier.

The observations therefore support the view that, for both upper and lower critical velocities, the relation $v_1/v_2 = \surd(\sigma_1/\sigma_2)$ is true.

But they do more than this. They suggest, if they do not fully prove, that if the surface tension had been constant throughout the measurements, in each case, the curves obtained by plotting l against $v/\surd\sigma$ would have been identical. It is therefore worth while to attempt an interpretation of this possibility.

15. *A Simple Relation Suggested by the Results of* § 14.—The relation between the variables, as given by Lord Rayleigh's theory, can be written in the form

$$l/d = v \cdot 1/y \cdot \log a_0/a \cdot \sqrt{(\rho d/\sigma)}.$$

Then, denoting the variables for two different solutions by the suffixes 1 and 2, respectively, we have

$$\frac{l_1}{d_1} = v_1 \left[\frac{1}{y} \log \frac{a_0}{a}\right]_1 \cdot \sqrt{\frac{\rho_1 d_1}{\sigma_1}}$$

and

$$\frac{l_2}{d_2} = v_2 \left[\frac{1}{y} \log \frac{a_0}{a}\right]_2 \cdot \sqrt{\frac{\rho_2 d_2}{\sigma_2}}$$

and if, as the results in § 14 suggest, we have always

$$l_1/d_1 = l_2/d_2, \quad \text{when } v_1 \sqrt{(\rho_1 d_1/\sigma_1)} = v_2 \sqrt{(\rho_2 d_2/\sigma_2)},$$

it follows at the same time that

$$\left[\frac{1}{y} \cdot \log \frac{a_0}{a}\right]_1 = \left[\frac{1}{y} \cdot \log \frac{a_0}{a}\right]_2.$$

From which, if y can be considered constant, we get the simple result that the initial disturbance is the same fraction of the disintegrating disturbance when the velocities v_1 and v_2 are in the ratio $\sqrt{(\sigma_1/\rho_1 d_1)} : \sqrt{(\sigma_2/\rho_2 d_2)}$.

16. *More Accurate Determination of the Relation between the Form of the lv Curve and the Diameter of the Jet. Verification of the Relation of* § 15.—The immediate object of the experiments (a) and (c) was to test whether the critical velocities are connected by the relation $v_1/v_2 = \sqrt{(d_2/d_1)}$, when σ is constant.

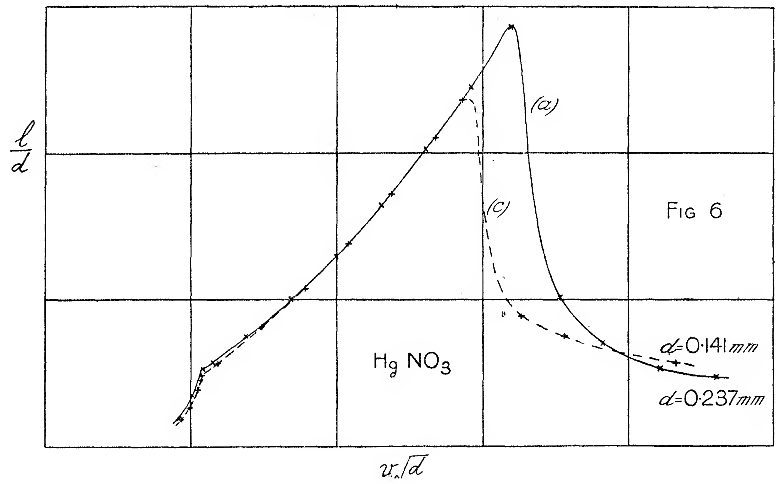
For this purpose it would have been sufficient to re-draw the curves (a) and (c) of fig. 3, keeping the ordinates the same but multiplying each abscissa by $\sqrt{d_1}$ in the one case and by $\sqrt{d_2}$ in the other. We made this change in the abscissæ, but, instead of keeping the ordinates the same as in fig. 3, we divided each by the diameter of the jet. We could then, in addition, see at a glance whether § 15 was confirmed.

The result is shown in fig. 6, where, therefore, the ordinates represent the values of l/d and the abscissæ those of $v\sqrt{d}$, for the two jets.

The curves are even more nearly alike than those of fig. 5. It is only in the region of the upper critical velocities that they differ appreciably.

In estimating how much importance to attach to this difference, it must be remembered that small uncontrollable differences in the experimental conditions are always present. It is impossible, for example, to construct a pair of nozzles identical in all respects except size. Small differences in shape, *e.g.* in the cylindricality of the orifice, influence the properties of the jet perceptibly and we think, from experience, that the differences shown

in fig. 6 are more than probably due to causes of this nature. Moreover, the results already given in § 6, for a series of jets, are evidence that, for



the upper critical velocity, the result $v_c \propto 1/\sqrt{d}$ is, in general, not far from true.

17. *Summary of Section I.*—We have performed a few supplementary experiments with jets of mercury and of other liquids falling in air, but before proceeding to them it will be convenient to summarize the results already obtained.

These results apply only to mercury jets of different diameters and surface tensions falling into aqueous solutions. The density (and viscosity) of the jet-fluid has remained unchanged.

(i) We have found that there are two critical velocities and that, for any pair of jets, corresponding critical velocities are connected by the relation

$$v_1 : v_2 = \sqrt{(\sigma_1/d_1)} : \sqrt{(\sigma_2/d_2)}.$$

(ii) We have taken the view that the critical velocities have their origin in certain known peculiarities of fluid motion which, we suppose, cause comparatively sudden changes, at particular velocities, in the initial amplitude of the disturbance under which the jet breaks down.

(iii) We have then found that the ratio of the initial to the final (disintegrating) disturbance is the same, at corresponding critical points, for all the jets examined.

(iv) The changes occurring at such critical points are thus always of exactly the same nature. Change of diameter or of surface tension merely alters the velocity of efflux at which each change occurs.

(v) We have found further that these regularities are not confined to the critical points. The sequence of changes of initial amplitude between these points is again always the same.

In consequence, the results can be correlated in the following simple way:—

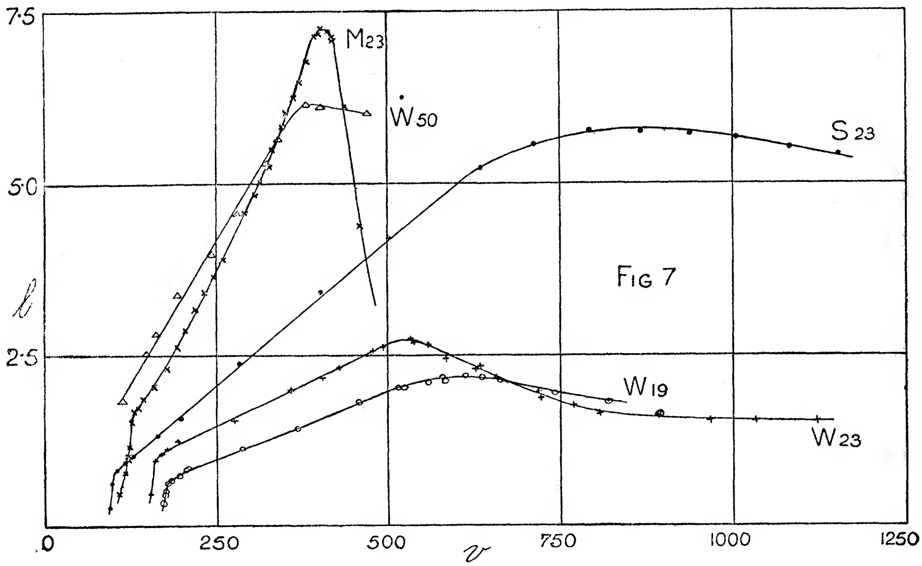
(vi) For any pair of jets, the initial disturbance is the same fraction of the final disturbance for any pair of velocities whose ratio is $\sqrt{(\sigma_1/d_1)} : \sqrt{(\sigma_2/d_2)}$, and, with these velocities, the jet-lengths are in the ratio

$$l_1 : l_2 = d_1 : d_2.$$

Section II.—*A Comparison with Jets of other Liquids.*

18. *Critical Velocities for Jets of Different Liquids formed in Air.*—Examination of the properties of jets of different liquids falling through air should afford an opportunity of testing our previous inference with respect to the influence of density. A comparison of mercury jets with water jets suggested itself not only because the densities of the two liquids are very different, but also because of Savart's results for water jets of comparatively large diameter, produced in a different way.

The time at our disposal did not permit many experiments. Fig. 7 shows one lv curve for mercury (M 23) and two (W 23 and W 19) for water. It also contains one lv curve for methylated spirit (S 23). The narrower nozzle (diam. 0·019 cm.) was used for the water only. The wider nozzle (diam. 0·023 cm.) was used for each of the three liquids.



The figure contains also an incomplete series of observations for water (W 50), with a nozzle of which the diameter was 0·0504 cm.

If the expression in § 8 is general, corresponding velocities should be in the ratio of the values of $\sqrt{(\sigma/\rho d)}$ for the different jets.

We took Quincke's values* of σ for the water and mercury. The values of σ and ρ for the methylated spirit were found by direct comparison with the water used.

The critical velocities v_1 and v_2, in the cases in which they were determined, are given in round numbers (from fig. 7) in the Table which follows:—

	d.	v_1.	v_2.	$v_1/\sqrt{(\sigma/\rho d)}$.	$v_2/\sqrt{(\sigma/\rho d)}$.
	cm.	cm./sec.	cm./sec.		
Water	0·02325	160	525	2·84	9·3
$\sigma = 74$	0·0186	180	600	2·85	9·5
$\rho = 1$					
Mercury	0·02325	125	405	3·0	9·7
$\sigma = 547$					
$\rho = 13·55$					
Methylated spirit ...	0·02325	100	(800)	2·80	(22·4)
$\sigma = 24·4$					
$\rho = 0·82$					

It will be seen from the numbers in the final columns that

$$v_c \propto (\sigma/\rho d)^{\frac{1}{2}}$$

is not far from true for the liquids water and mercury, and also for methylated spirit when the lower critical velocity is considered. In the latter case, however, the upper critical velocity is far too high. Instead of being in the neighbourhood of 350 cm. per second, as it should be to agree with the results for the other liquids, its value is about 800 cm. per second. We have not investigated this discrepancy sufficiently to be able to express a final opinion as to its cause. We found, however, that the viscosity of the methylated spirit was 43 per cent. greater than that of the water. In contrast with this, the viscosities of water and mercury are nearly equal. Further experiment would, no doubt, define the extent to which the critical velocity is affected by the viscosity of the liquid (*cf.* § 8).

19. *Analysis of Water Jet Results.*—It was now of interest to compare the curves of fig. 7 by the method of § 16.

In order to examine the relation between the water curves, it would have sufficed to plot the values of l/d against those of $v\sqrt{d}$; but, to permit a

* 'Ann. d. Phys.,' vol. 52, p. 1 (1894).

wider comparison, we have plotted the values of l/d against those of $v\sqrt{(\rho d/\sigma)}$ for each case of § 18. The result is shown in fig. 8.

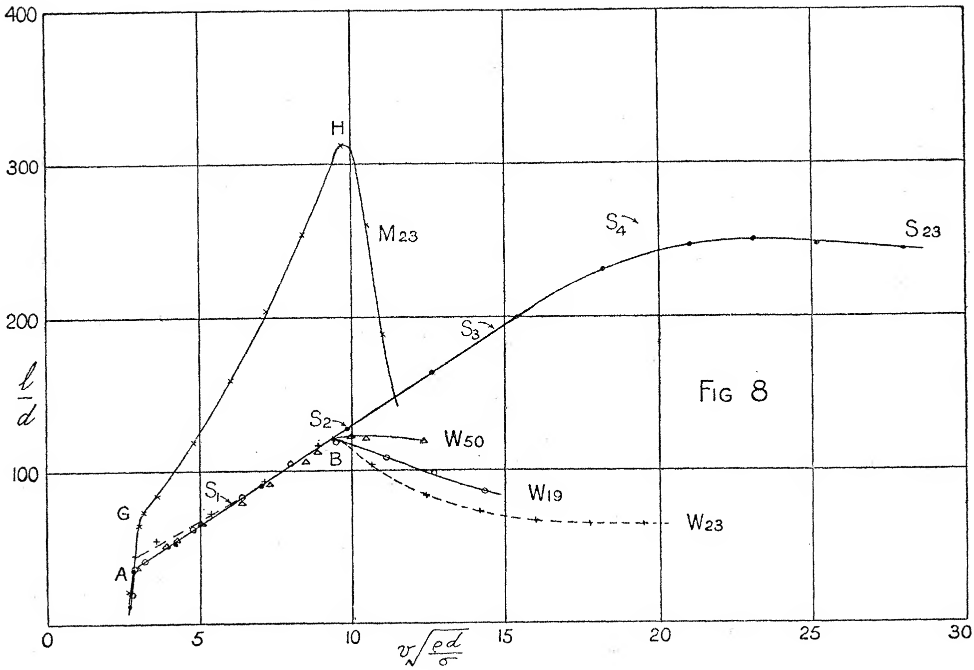


FIG 8

It will be seen that the water curves, instead of being widely separated as in fig. 7, are now practically coincident in the region (marked AB) between the critical points. The agreement between the W_{19} points (marked by ⊙) and the W_{50} points (marked by △) is very close. They lie, very nearly, upon a straight line passing through the origin. Except at the lower values of v, where the agreement is not very good,* the W_{23} points (marked by +) lie on the same straight line as the others.

Thus the relation outlined in § 15 appears again to hold; but, between the two critical points, it can now be expressed even more simply than before.

This happens because, if we assume that the portions of the curves lying between the two critical points are on a straight line passing through the origin and if, as in § 15, we have

$$l/d = v \,.\, 1/y \,.\, \log a_0/a \,.\, \sqrt{(\rho d/\sigma)},$$

* It should be mentioned that, under the experimental conditions, the length of a water jet is less easy to fix than that of a mercury jet. The end of the jet is more clearly visible in the latter case than in the former. The wetting of the glass by the water also causes trouble at the lower velocities.

it follows that the quantity

$$\frac{1}{y}\log\frac{a_0}{a} = \frac{l}{d} \Big/ v\sqrt{\frac{\rho d}{\sigma}} = \tan\theta,$$

where θ is the slope of the line AB, is constant for all the jets.

The simplest interpretation of this is (y constant) that a_0/a is constant and that therefore the initial amplitude of the disintegrating disturbance is the same fraction of the final disturbance, for all the jets, for all velocities lying between the limits specified.

Fig. 8 shows also another interesting result. The methylated spirit points (marked by ·) lie almost exactly on the same line (AB) as the water points. The initial amplitude is thus not only a constant fraction of the final amplitude as with water; but, in addition, this fraction has the same value for both liquids.

Without making a quantitative comparison it can be seen at once, from fig. 8, that, besides being variable, the initial amplitudes are much smaller fractions of the final amplitude when mercury is used.

These differences between mercury and the other liquids must of course have a cause. Possibly they are connected with the fact that mercury does not wet glass. It may be, also, that the magnitude of the initial disturbance is influenced by pulsations of pressure, occurring within the jet as each drop forms and breaks away, whose effects (owing, for example, to differences of density) are not of the same relative importance in each case.

In some approximate experiments with orifices of the kind referred to below (§ 22) we found that, although the jet-lengths for mercury were (as above) longer than the corresponding ones for water, the lv curves for the former liquid were now, as nearly as we could decide, parts of straight lines passing through the origin. With such orifices it therefore appears that, as with other liquids, the initial amplitude for mercury is now a constant fraction of the final amplitude.

20. *Numerical Comparisons between Water Jets and Mercury Jets.*—If we assume the value of y to be that calculable from Lord Rayleigh's theory, viz.,

$$y = \sqrt{8} \times 0{\cdot}343 = 0{\cdot}97,$$

we can compare the initial amplitudes quantitatively.

Thus, from the water curves, we get very nearly $l/d \div v\sqrt{(\rho d/\sigma)} = 13$, and hence $a_0/a = e^{12{\cdot}6}$. Similarly, from the mercury curve, we get at G (the lower critical velocity) $a_0/a = e^{22{\cdot}8}$ and at H (the higher critical velocity) $a_0/a = e^{31{\cdot}7}$. Whence we get that the initial amplitude between A and B (water and methylated spirit) is, in round numbers, 20,000 times as great as that at G (mercury), while that at G is about 8,000 times as great as that at H.

21. *Confirmation of Lord Rayleigh's Theory of Jets.*—Lord Rayleigh has pointed out* that some of Savart's results can be explained on the assumption that "the disturbances remain always† of the same character, so that the *time* of disintegration is constant."

This suggestion with respect to Savart's jets, that the time of disintegration is constant, is clearly true for our water jets between A and B (fig. 8). We have measured both l and v (fig. 7) and have found that the ratio $l/v = t$ is constant.

Another consequence of the above assumption, pointed out by Lord Rayleigh, but apparently at variance with Savart's results, is that the jet-length for a given velocity of efflux should be proportional to $d^{3/2}$. This inference is, however, again in agreement with our results. For example, when the velocity is 500 cm./sec., it will be seen from fig. 7 that the jet-lengths are in the ratio 1·38 : 1, while $(d_1/d_2)^{3/2} = 1{\cdot}40$.

If the data upon which Savart bases his conclusion (that the jet-length is proportional to the diameter) be examined, it will be found that the evidence for the conclusion is very slight. It is contained in two series of results‡ for orifices of which the diameters were 3 and 6 mm. respectively.

Savart himself remarks that the difficulty of obtaining definite measurements increases with the diameter of the jet. The jet-length becomes increasingly dependent upon extraneous disturbances until, as he points out, it is impossible to obtain even approximate measurements when the diameter of the orifice is 9 mm. Even with the narrower jets used by him, these effects are considerable and it will be found that only one of his sets of results for the 6 mm. orifice agrees reasonably with his first "law"—that the jet-length is proportional to the velocity of the efflux. While, when all his results with this orifice are considered, they furnish as little support for his second "law" as for that which we have found to be true for the jets we have used.

It is interesting to compare Savart's results with his narrower jet (diameter 0·3 cm.), which were comparatively regular, with our data.

For this purpose we may take his "fourth series,"§ in which the jet was protected as far as possible from outside disturbances. If, for this set, l/d is plotted against $v\sqrt{(\rho d/\sigma)}$ we get a straight line passing through the origin which, curiously enough, coincides almost exactly with the line AB. This relationship is shown in fig. 8, where the heads of the arrows marked

* 'Theory of Sound,' vol. 2, p. 363 (1896).

† *I.e.*, for different velocities of efflux.

‡ 'Ann. Chim. Phys.,' vol. 53, p. 368.

§ *Loc. cit.*, p. 368.

S_1, S_2, S_3, S_4, represent Savart's results treated in this way. It should be remarked, however, that in this case d, the diameter of the orifice, is not, even approximately,* the mean diameter of the jet. For equal velocities, this jet was more than 60 times as long as the narrowest jet we used and more than 15 times as long as the widest.

Lord Rayleigh, referring to the discrepancy between Savart's results and theory, remarks that " it may well be doubted whether the length of the continuous portion obeys any very simple laws, even when external disturbances are avoided as far as possible." We have therefore thought it of interest to point out that conditions can be found under which the discrepancy between theory and experiment disappears.

22. *Dependence of the Critical Velocities upon the Shape of the Nozzle.*—There is one respect in which Savart's results differ very noticeably from ours. They show no evidence of critical velocities. The jet-length continues to increase long after the velocity reaches the value above which, if his jets had behaved like ours, a shortening would have ensued.

It is easy to show experimentally that this difference is due to the difference between the orifices. We made an approximate series of observations with an orifice of about 0·025 cm. diameter, similar in character to those used by Savart. It was constructed by boring a hole, bevelled from the outside, in a thin steel plate. This plate was fixed horizontally, at the end of the supply tube, in the position previously occupied in turn by each of the glass nozzles already described. We found for water with this orifice that, as in Savart's experiments, the jet-length continued to increase until we reached the greatest head we could apply conveniently. The velocity of efflux was then more than three times as great as that which, with a glass nozzle of the same diameter, would have been the upper critical velocity.

23. *The Origin of the Critical Velocities.*—The results just described throw light upon the cause of the phenomena we have observed. They show that the conditions of escape of the fluid may be such that the upper critical velocity either does not exist or is relatively very high and therefore invite a comparison between one kind of orifice and the other. This comparison can be made, very roughly, as follows :—

When the liquid leaves the orifice of a nozzle such as we have used, assuming that " turbulence " has not arisen within the tube, the stream-lines will be practically parallel. Owing to viscosity, however, the velocity will be greater at the centre of the jet than it is near the sides. This difference of velocities will not be maintained after the liquid leaves the orifice, for the

* *Loc. cit.*, p. 370.

force exerted upon the moving liquid by the walls of the nozzle will disappear from each element of the jet as it forms. But viscous forces will continue to operate between the more rapidly moving central part of the element and the less rapidly moving outer parts until the differences between their velocities are destroyed.

Since the velocity along the axis, neglecting the effects of gravity, must thus tend to fall after the liquid leaves the nozzle, the axial stream-lines must tend to broaden and, while this is occurring, the velocity along these stream-lines will have a radial component from the centre outwards. It is while the liquid particles are executing the movements necessary at this stage of the jet's existence that stream-line motion seems most likely to break down. We may suppose, in fact, that excessive radial velocity is a precursor of turbulence and may expect the appearance of the latter to be controlled by the means at disposal for the conversion of radial kinetic energy into other forms. To express this view in another way, imagine an element of the jet of length dl to have its radius increased from r to $r+dr$ by the influx at its centre of a volume $d\mathrm{V}$ of liquid possessing kinetic energy, of which the amount $\alpha\rho v^2 d\mathrm{V}$ may be regarded as radial and tending to degenerate into energy of turbulence. The relation between $d\mathrm{V}$ and dr is $d\mathrm{V} = dl \,.\, 2\pi r\, dr$; that between dr and $d\mathrm{S}$, the increase in the surface of the element, is $d\mathrm{S} = dl \,.\, 2\pi\, dr$. Hence the increase of surface energy corresponding with the influx of $d\mathrm{V}$ is $\sigma d\mathrm{S} = \sigma d\mathrm{V}/r$. And if $\sigma d\mathrm{V}/r$ exceeds $\alpha\rho v^2 d\mathrm{V}$, *i.e.*, if $\sigma/\rho r$ exceeds αv^2, the increase of surface energy would be more than sufficient to replace the whole of the radial kinetic energy from which turbulence might arise and so could prevent the degeneration of energy which otherwise might ensue.

Such considerations indicate, perhaps more clearly than those of § 8, why it is that, with our nozzles, there are critical velocities which depend upon the relative values of $\sigma/\rho r$ and of v^2.

In the case of an orifice like that used by Savart, the region within which there are appreciable differences of velocity, between contiguous moving elements of the fluid, is of restricted extent and the effects of viscosity are reduced to a minimum. In this case the stream-lines converge as the liquid leaves the orifice. The velocity is greatest at the sides; but the pressure is greatest at the centre. The hydrodynamic forces operate to remove both these differences, and at a short distance below the orifice (gravity neglected) the stream-lines become parallel, and the velocity becomes uniform, across the jet. The cause of turbulence present in the first case has disappeared.

24. *Summary of Section II.*—The experiments of §§ 18 to 23 may be

described as forming a link between the experiments of §§ 1 to 16 and the well-known experiments of Savart.

Nozzles of the same kind as before were used, but the jet liquid was now surrounded by air. Using mercury and water, the relation $v_c \propto (\sigma/\rho d)^{\frac{1}{2}}$ was tested and found to be nearly true.

The mercury jet had similar properties to those previously studied. As before, between the critical velocities, dl/dv increases as v increases.

Water behaves differently. Jets of diameters between 0·18 and 0·5 mm. were examined. Their properties may be formulated thus:

(i) Between the critical points, each lv curve forms part of a straight line passing through the origin. Hence for each jet

$$dl/dv = l/v = t = \text{a constant},$$

where t is the time of disintegration, *i.e.*, the time taken by each element of the liquid to pass from the top to the bottom of the jet.

(ii) The value of this constant increases with the diameter of the jet, but when l/d is plotted against $v\sqrt{(\rho d/\sigma)}$ the observations for all the jets fall upon the same straight line.

(iii) From this it follows that

$$l/d \div v\sqrt{(\rho d/\sigma)} = l/v \,.\, \sqrt{(\sigma/\rho d^3)} = t\sqrt{(\sigma/\rho d^3)}$$

has the same value for all the jets.

(iv) The cause of disintegration is taken to be the disturbance which originates at the orifice.

(v) Lord Rayleigh's theory leads to the result that the rate of growth of this disturbance is given by

$$ae^{k\tau\sqrt{(\sigma/\rho d^3)}}$$

where a is the initial amplitude, k is a constant (§ 9), τ is the time which has elapsed since the disturbance began.

(vi) If this theory is applicable, the ratio of the initial to the final amplitude for each jet is given by

$$e^{-kt\sqrt{(\sigma/\rho d^3)}}$$

and is therefore the same for all, since we have found $t\sqrt{\sigma/\rho d^3}$ to be constant. The initial amplitude is, apparently, a constant fraction of the diameter of the jet.

(vii) As Lord Rayleigh has pointed out, some of Savart's results suggest a simple relation of this kind, while others suggest that the relation is more complicated.

(viii) The present experiments indicate that if narrower jets had been

used by Savart, the simple relation would have sufficed to define their behaviour.

(ix) The critical velocities, if they exist, obey a different law when, as in Savart's experiments, the liquid escapes through a sharp-edged hole in a horizontal plate. Our critical velocities owe their existence to the shape of the nozzle we have used. This favours turbulence—eddies or vortices—as indicated in the text.

---

*On Residual Magnetism in Relation to Magnetic Shielding.**

By ERNEST WILSON, M.Inst.C.E., M.I.E.E., Siemens Professor of Electrical Engineering, and J. W. NICHOLSON, M.A., D.Sc., Professor of Mathematics in the University of London.

(Communicated by Prof. J. A. Fleming, F.R.S. Received February 17, 1917.)

The present communication records the final stages of an investigation, of a somewhat exhaustive character, into the various problems which were presented by the necessity for constructing a magnetic shield suitable for large spaces, and capable of giving such a degree of shielding that the internal field caused by the earth should not exceed the order of magnitude 10^{-3} C.G.S. unit. The theoretical calculation of the best form of shield, and the details of its construction, were given, together with an examination of the various methods of testing the efficiencies of large shields, in the first paper.† A study of the effect of leakage through small air spaces was made at the same time, and it became apparent that not only this problem, but several others which are vital to the production of the theoretical efficiency of a shield, needed a more careful study than they had received hitherto.

The usual process adopted for removing permanent magnetism from a shell or set of shells, by the reversals of a slowly decreasing current, ceases to be efficient when the magnetisation is very small, unless special methods are introduced, and there was previously no definite indication of the degree of accuracy with which the magnetic induction at any point due to a coil, wound in a helix on one member of a set of spherical shells, could be either predicted or measured. A study of these problems was made, and the results described in a second paper.‡

* This investigation has again been facilitated by a grant made by the Council of the Society out of the Gore Fund.

† 'Roy. Soc. Proc.,' A, vol. 92, p. 529 (1916).

‡ 'Roy. Soc. Proc.,' A, vol. 93, p. 129 (1917).